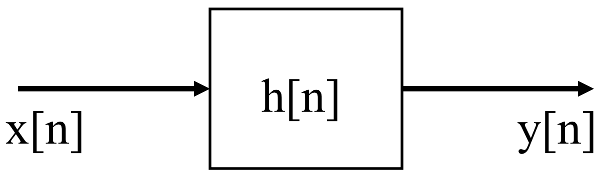
Applications of Z-Transform in Signal Processing - II

Yogananda Isukapalli

Frequency response estimation



The frequency response of a system can be obtained from its z-transform as:

Given {h[n]},

We have:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n].z^{-n}$$

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\theta n}$$

Implies that :

$$H(e^{j\theta}) = H(z)\Big|_{z=e^{j\theta}} \qquad (1)$$

Example : A Comb Filter

$$y[n] = x[n] - x[n-12]$$

$$H(z) = 1 - z^{-12} = \frac{z^{12} - 1}{z^{12}}$$

zeros $z_m = 1e^{jm(2\pi/12)}$ $_{m=0,\ 1,\ 2,\ \ldots 11}$

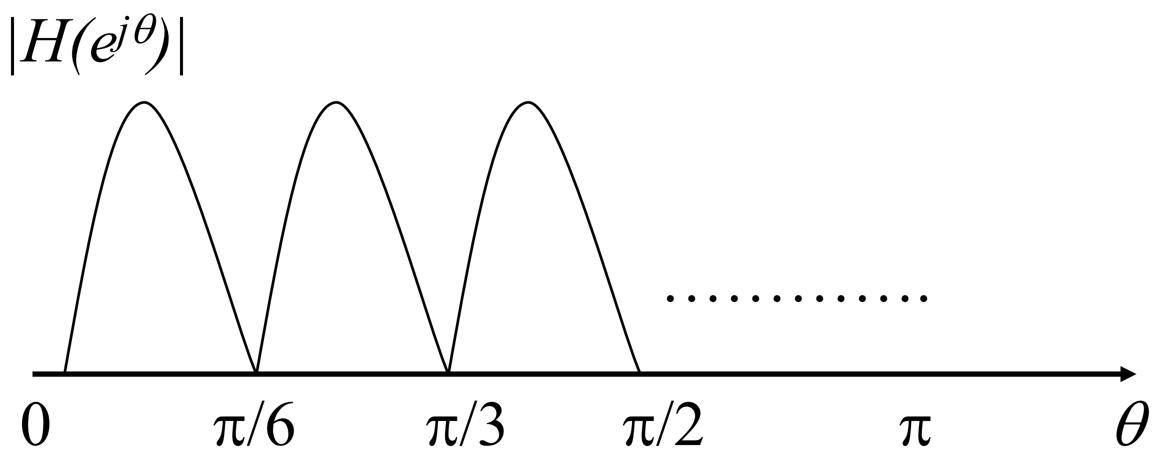
12 zeros and 12 poles

$$H(e^{j\theta}) = 1 - e^{-j12\theta}$$

$$= j2\, e^{-j6\theta}\sin(6\theta)$$

$$= 2\sin(6\theta)e^{j(\pi/2 - 6\theta)}$$

Plot of $H(e^{j\theta})$



Frequency response function properties :

(1) $H(e^{j\theta})$ is a periodic function.

$$H(e^{j(\theta+2\pi)}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j(\theta+2\pi)m}$$

$$H(e^{j(\theta+2\pi)}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j\theta m}.e^{-j2\pi m}$$

$e^{-j2\pi m} = 1$ for all integer values of m

$Therefore:$

$$H(e^{j(\theta+2\pi)}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j\theta m}$$

$$H(e^{j(\theta+2\pi)}) = H(e^{j\theta})$$

(2) $M = |H(e^{j\theta})| = |H(e^{-j\theta})|$

$$P = \langle H(e^{j\theta}) = -\langle H(e^{-j\theta})$$

(3) $M_{dB} = 20log_{10}M$

Frequency units used in discrete-time systems:

- Two frequency units are normally used to describe the frequency response of discrete-time systems: ω (rad s^{-1}) and f (Hz)
- The frequency response is found by letting

$$z = e^{j\theta} = e^{j\omega T} = e^{j2\pi fT}$$

and then evaluating the z- transfer function, H(z), in the following intervals:

$0 \leqslant \omega \leqslant \omega_s/2$	rad s^{-1}
$0 \leqslant \omega \leqslant \pi/T$	rad s^{-1}
$0 \leqslant \omega \leqslant \pi$	(normalized)
$0 \leqslant f \leqslant F_s/2$	Hz
$0 \leqslant f \leqslant 1/2T$	Hz
$0 \leqslant f \leqslant 1/2$	(normalized)

- The following figure shows how ωT and z change as ω varies from 0 to ω_s.

- As the angle $\theta = \omega T$ goes from 0 to 2π the value of z varies from 1 through j and back to 1.

f (Hz)	ω (rad s^{-1})	$\theta = \omega T$ (rad)	$z = e^{j\omega T}$
0	0	0	1
$\frac{F_s}{8}$	$\frac{\omega_s}{8}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$
$\frac{F_s}{4}$	$\frac{\omega_s}{4}$	$\frac{\pi}{2}$	j
$\frac{3F_s}{8}$	$\frac{3\omega_s}{8}$	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$
$\frac{F_s}{2}$	$\frac{\omega_s}{2}$	π	-1
$\frac{5F_s}{8}$	$\frac{5\omega_s}{8}$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$
$\frac{3F_s}{4}$	$\frac{3\omega_s}{4}$	$\frac{3\pi}{2}$	$-j$
$\frac{7F_s}{8}$	$\frac{7\omega_s}{8}$	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$
F_s	ω_s	2π	1

$F_s = 1/T$ is the sampling frequency in Hz; T is the sampling period; $\omega_s = 2\pi/T$ is the sampling frequency in rad s^{-1}.

Fig: Units of frequency used in discrete-time systems and their relationship to points on the unit circle

• The following figure also makes it clear that the frequency response of a discrete-time system is cyclic.

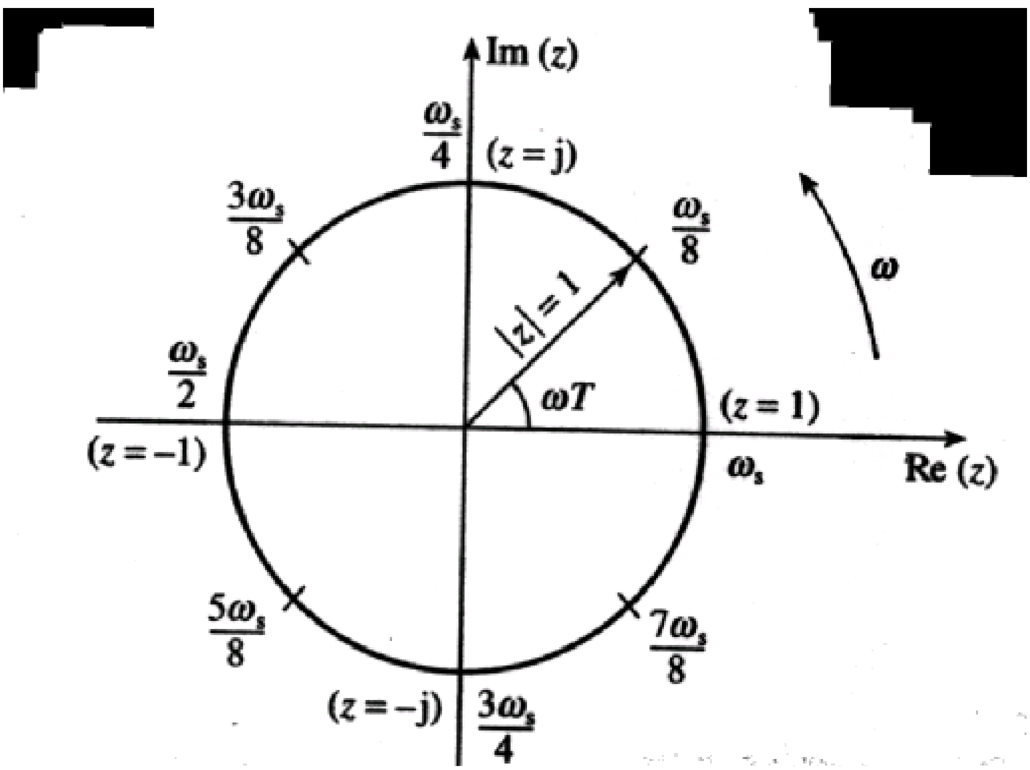


Fig: z-plane unit circle showing critical frequency points

• As we go round the circle one or more revolutions the values of z simply repeat.

Example)

Given the frequency response specification for a bandpass discrete-time filter in Hz as

passband	6–10 kHz
stopbands	0–4 kHz and 12–16 kHz
sampling frequency	32 kHz

(1) express the specifications in normalized frequency, f,

(2) convert the specification from standard units of Hz to rad s^{-1}, and

(3) convert the specifications from the units of rad s^{-1} in part (2) to normalized frequency, ω.

Soln)

(1) The bandedge frequencies, which are in units of Hz, can be expressed in normalized form by simply dividing each frequency by the sampling frequency. Thus, the specification in normalized form becomes

passband	0.1875–0.3125
stopbands	0–0.125 and 0.375–0.5
sampling frequency	1

(2) Since $\omega = 2\pi f$, each bandedge frequency is simply multiplied by 2π to convert it into rad s^{-1}. The frequency response specifications now become

passband	$12\,000\pi$–$20\,000\pi$ rad s^{-1}
stopbands	0–8000π and $24\,000\pi$–$32\,000\pi$ rad s^{-1}
sampling frequency	$64\,000\pi$ rad s^{-1}

(3) The bandedge frequencies in (2) can be expressed in normalized form by dividing each frequency by 32 kHz (the sampling frequency), for example

$$12\,000\pi \rightarrow \frac{12\,000\pi}{32\,000} = \frac{3\pi}{8}$$

Thus the specifications become

passband	$3\pi/8$–$5\pi/8$
stopbands	0–$\pi/4$ and $3\pi/4$–π
sampling frequency	2π

Geometric evaluation of frequency response:

We have

$$H(z) = \frac{K(z-z_1)(z-z_2)...(z-z_N)}{(z-p_1)(z-p_2)...(z-p_N)} = \frac{\prod_{i=1}^{N} K(z-z_i)}{\prod_{i=1}^{N}(z-p_i)} \qquad (2)$$

The frequency response is given by

$$H(e^{j\theta}) = H(e^{j\omega T}) = \frac{\prod_{i=1}^{N} K(e^{j\omega T}-z_i)}{\prod_{i=1}^{N}(e^{j\omega T}-p_i)} \quad \text{for } 0 \le \omega \le \omega_s/2 \qquad (3)$$

• For a z-transform with only two zeros and two poles, the frequency response is given by

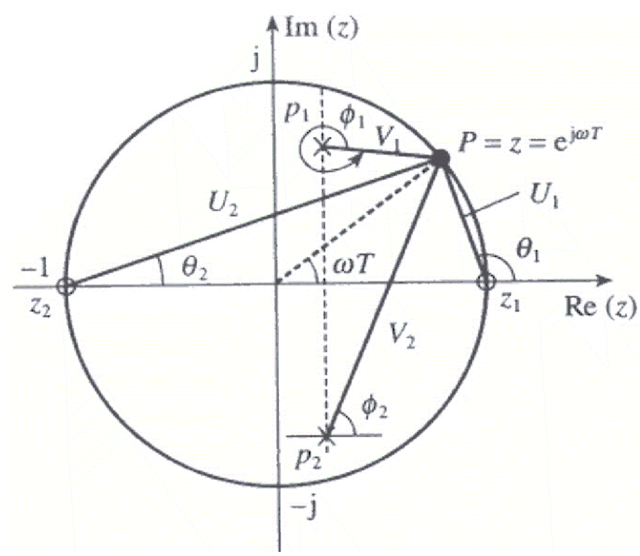
$$H(e^{j\theta}) = H(e^{j\omega T}) = \frac{K(e^{j\omega T}-z_1)(e^{j\omega T}-z_2)}{(e^{j\omega T}-p_1)(e^{j\omega T}-p_2)}$$

$$= \frac{KU_1\angle\theta_1 U_2\angle\theta_2}{V_1\angle\phi_1 V_2\angle\phi_2} \qquad (4)$$

where U_1 and U_2 ⟶ distances from the zeros to the point $z=e^{j\omega T}$

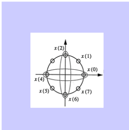
V_1 and V_2 ⟶ distances from the poles to the point $z=e^{j\omega T}$

Fig: Geometric evaluation of the frequency response from pole-zero diagram



- $$\left|H(e^{j\omega T})\right| = \frac{U_1 U_2}{V_1 V_2}, \quad K = 1 \qquad (5)$$

$$\angle\left[H(e^{j\omega T})\right] = \theta_1 + \theta_2 - (\phi_1 + \phi_2)$$



- In general, in the geometric method, the frequency response at a given frequency ω (at an angle ωT) is given by

$$\frac{U_i \angle \theta_i}{V_j \angle \phi_j}, \quad i = 1, \ldots \text{ number of zeros}, \quad j = 1, \ldots \text{ number of poles}$$

Example: Determine the frequency response at dc, 1/8, 1/4, 3/8 and 1/2 the sampling frequency of the causal discrete-time system with the following z-transform:

$$H(z) = \frac{z+1}{z-0.7071}$$

Soln) H(z) has single pole and single zero. So,

$$H(e^{j\omega T}) = \frac{U\angle\theta}{V\angle\phi} = \frac{e^{j\omega T}+1}{e^{j\omega T}-0.7071} = \frac{1+\cos(\omega T)+j\sin(\omega T)}{\cos(\omega T)-0.7071+j\sin(\omega T)} \quad (6)$$

At dc, ωT=0 and the zero and pole vectors to the point z=0 are $2\angle 0^{\circ}$ and $0.2929\angle 0^{\circ}$

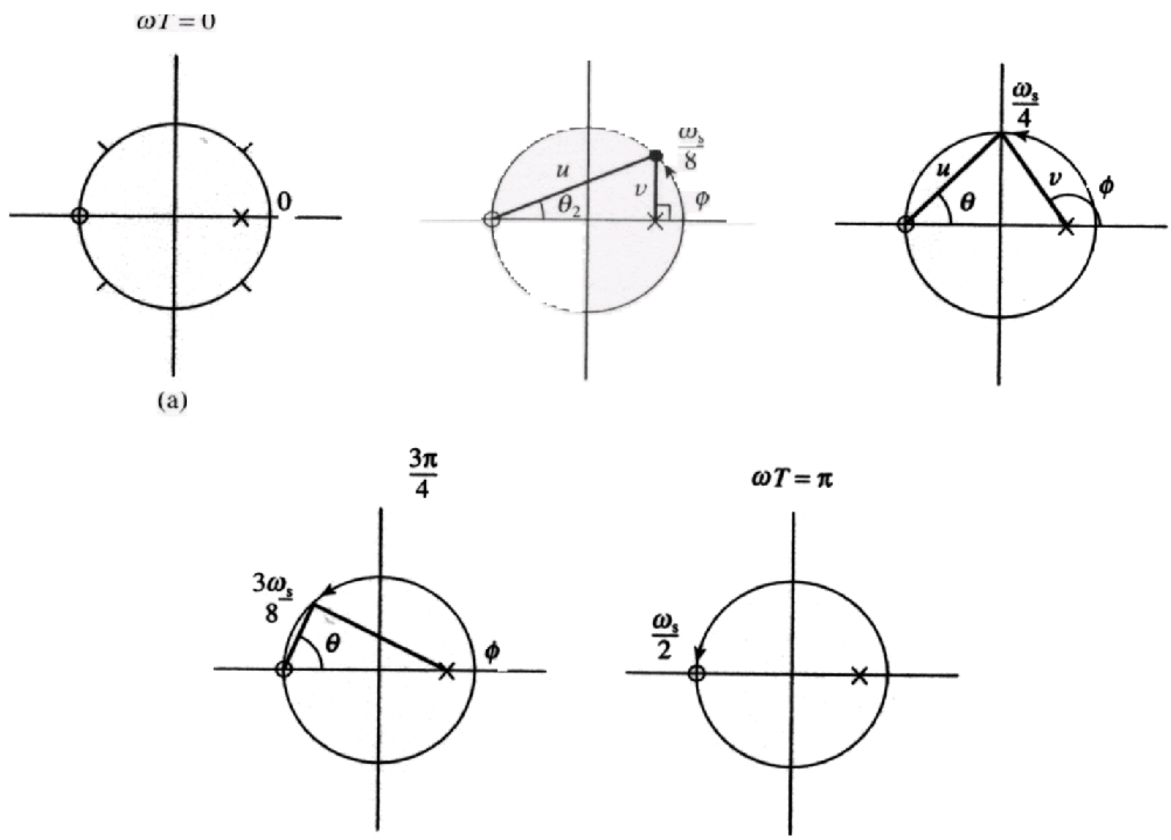


Fig: Frequency response estimation using geometric method and pole-zero diagram

Thus the frequency response is given by

$$H(e^{j\omega T}) = 2/0.2929 = 6.828\angle 0^{o}$$

At $\omega = \omega_s/8, \omega T = \omega_s/8F_s = \pi/4$

$$H(e^{j\omega T}) = \frac{1+\cos(\pi/4)+j\sin(\pi/4)}{\cos(\pi/4)-0.7071+j\sin(\pi/4)}$$

$$= \frac{1.8477\angle 22.5^{o}}{0.7071\angle 90^{o}} = 2.6131\angle -67.5^{o}$$

The responses at the remaining frequencies are summarized below:

ω (rad s^{-1})	$\theta = \omega T$ (rad)	$\lvert H(e^{j\omega T}) \rvert$	$\angle H(e^{j\omega T})$ (degrees)
0	0	6.828	0
$\omega_s/8$	$\pi/4$	2.6131	−67.5
$\omega_s/4$	$\pi/2$	1.1547	−80.26
$3\omega_s/8$	$3\pi/4$	0.4840	−85.93
$\omega_s/2$	π	0	0

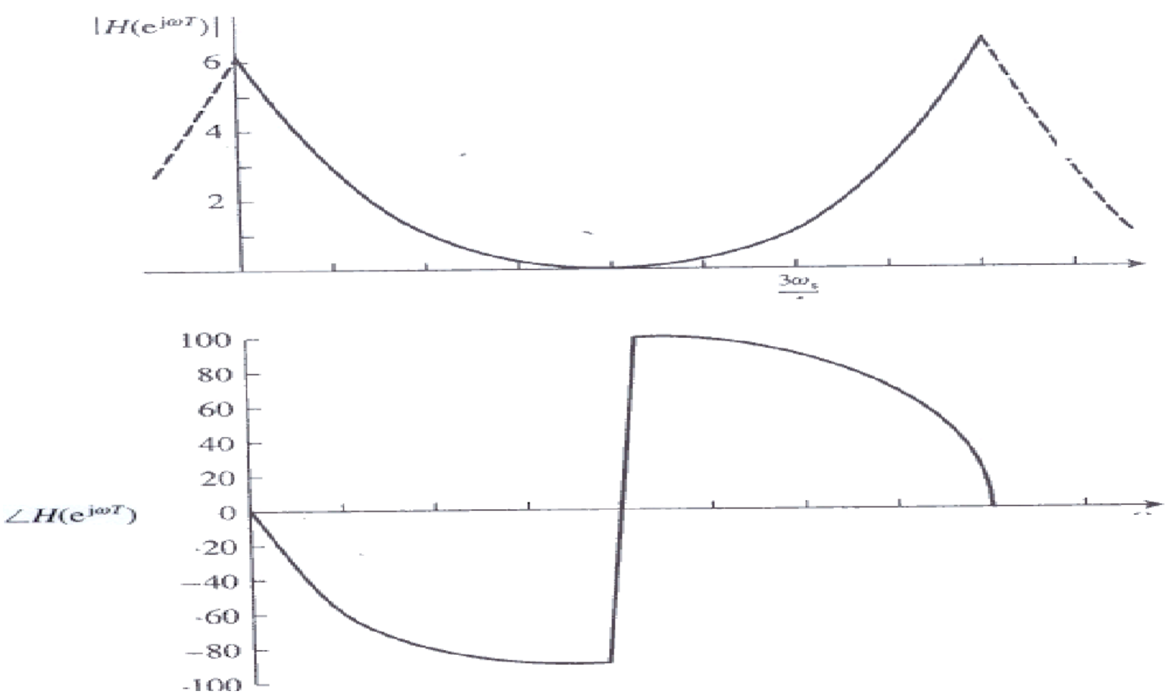


Fig: A sketch of the frequency response

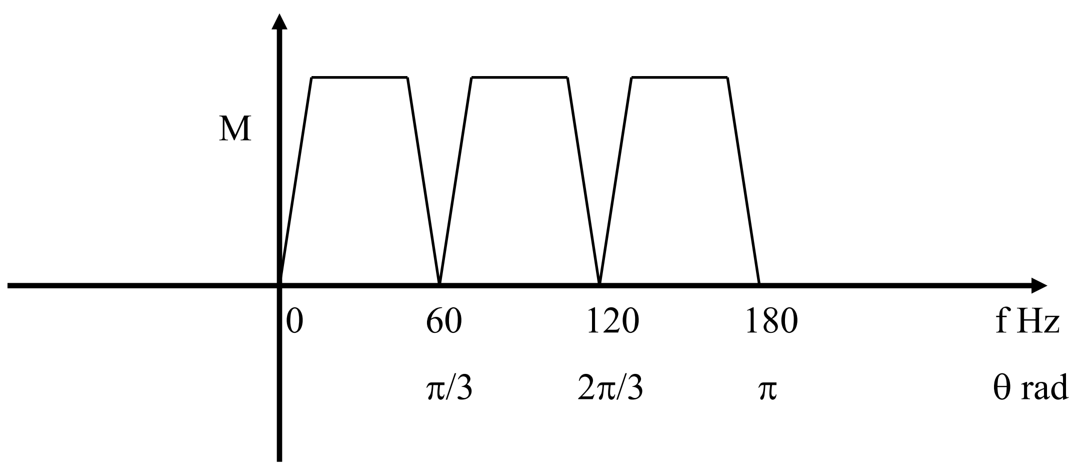
- It is observed that the magnitude response is symmetrical about half the sampling frequency (Nyquist frequency), and the phase response antisymmetrical about the same frequency.
- The frequency response is periodic with a period of ω_s.

Example : The Graphical Design of a Comb filter :

- In medical applications, the 60Hz frequency of the power supply is often "picked up" by the test equipment (EKG recorder)
- Also harmonically related frequencies such as $f_2 = 2x60 = 120Hz$, and $f_3 = 3x60 = 180$ Hz are generated because of non-linear phenomena.

• The object of a digital filter design is to eliminate or suppress these unwanted frequencies which distort or mask up the signals of interest.

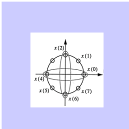
Thus the desired response of the filter would be :



Thus we require a nonrecursive (FIR) comb filter:

Note : Digital Frequencies

Sampling frequency 360 Hz



Thus we have :

$\theta_1 = \omega_1 T = 2\pi(60)/360 = \pi/3$

$\theta_2 = \omega_2 T = 2\pi(120)/360 = 2\pi/3$

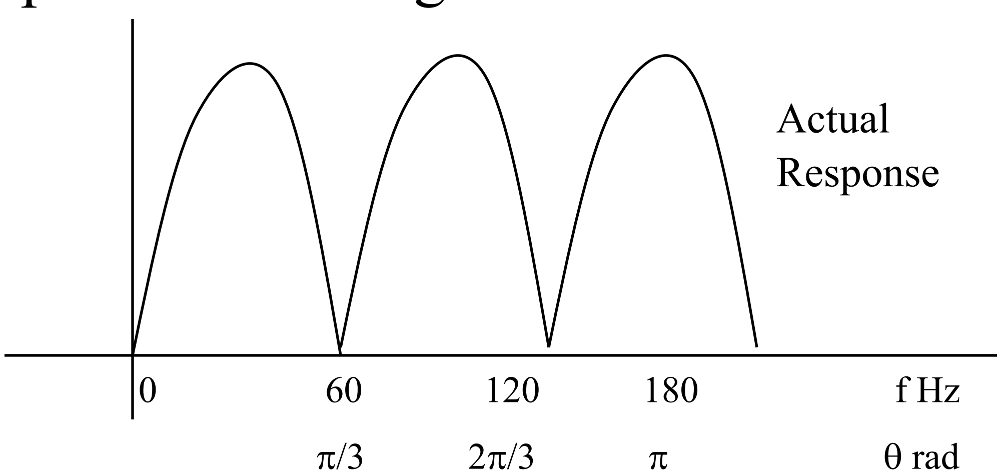
$\theta_3 = \omega_3 T = 2\pi(180)/360 = \pi$

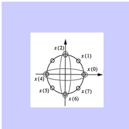
Proposed pole - zero design :

Note :

1.) Complex zeros must occur in conjugate pairs.

2.) $\theta = 0$ is added to eliminate any DC component in the signal.





$$H(z) = (z-1)(z-e^{j\frac{\pi}{3}})(z-e^{j\frac{2\pi}{3}})(z-e^{j\pi})(z-e^{j\frac{4\pi}{3}})(z-e^{j\frac{5\pi}{3}})$$

$$H(z) = z^6 - 1$$

Implies :

$$y[n] = x[n+6] - x[n]$$

But the above obtained filter is non-causal !!
To make it causal filter we place six poles at z = 0.

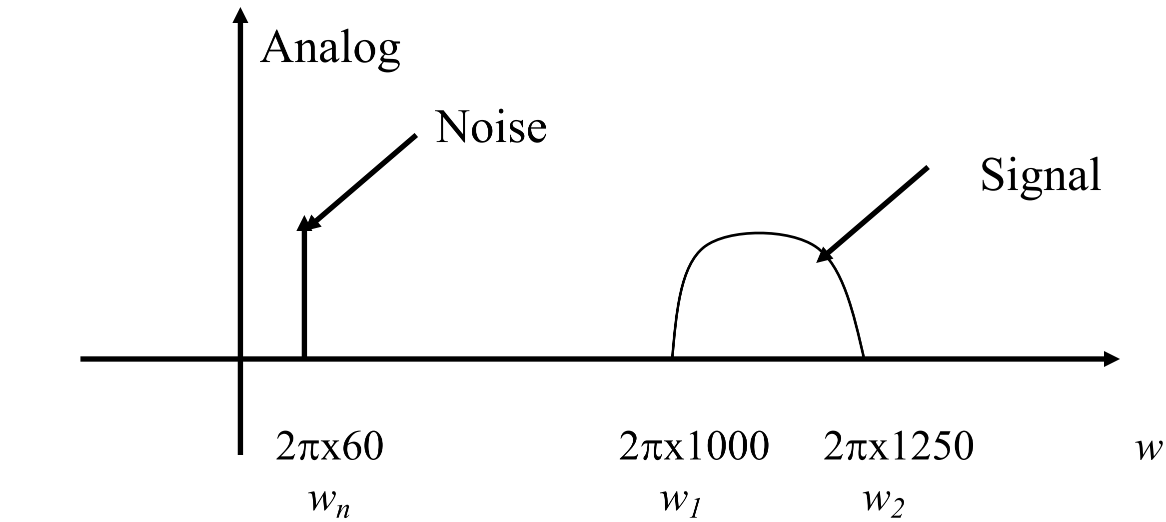
$$H(z) = \frac{z^6 - 1}{z^6}$$

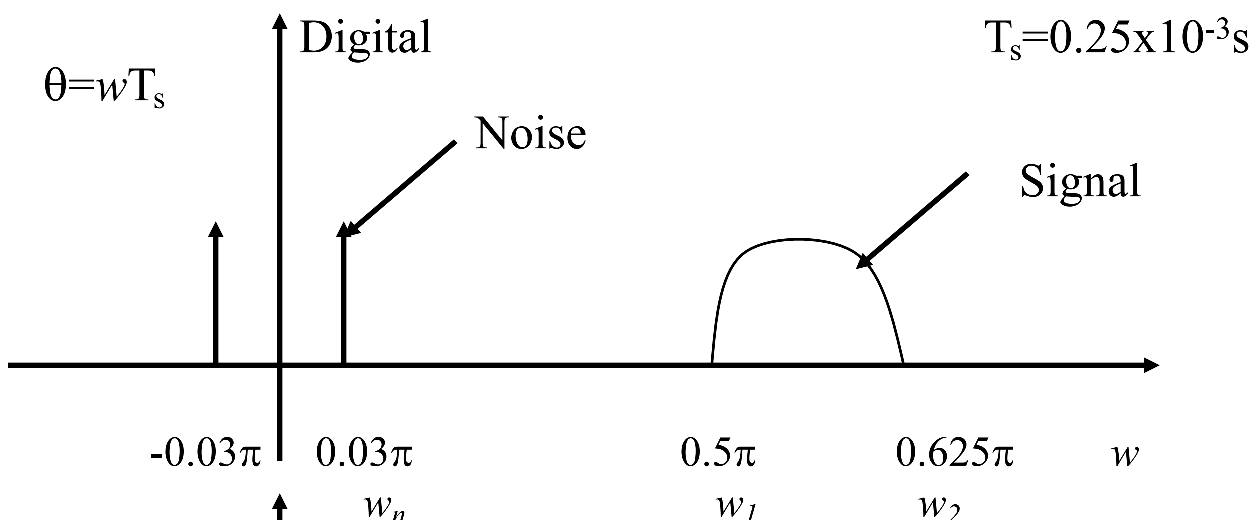
$$\therefore \Rightarrow H(z) = (1 - z^{-6})$$

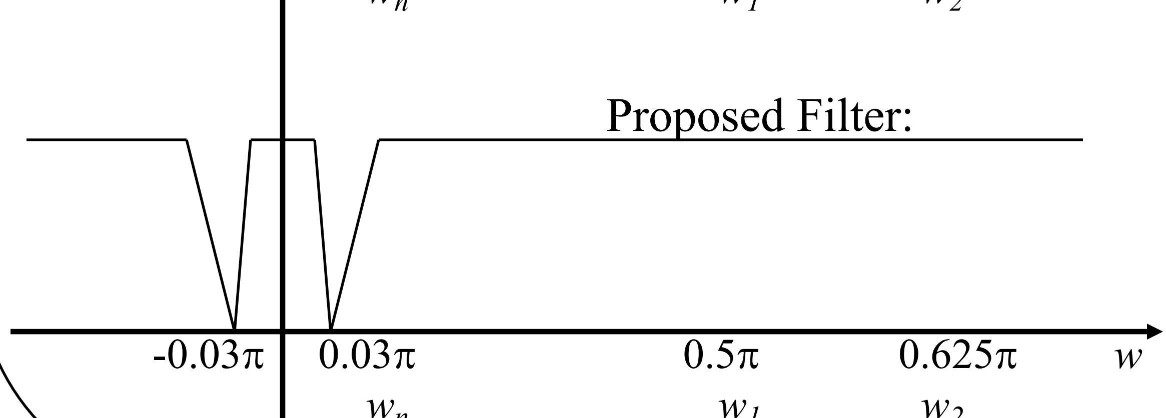
Thus the required causal FIR comb filter is:

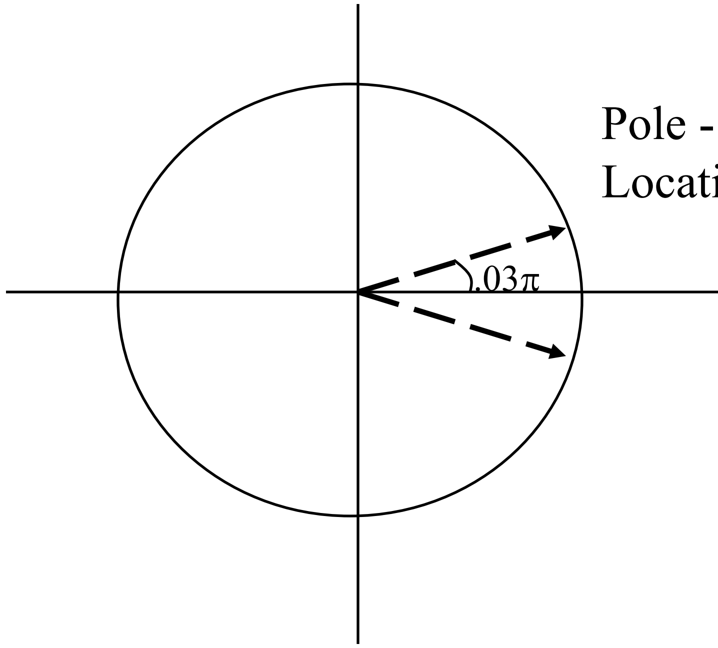
$$y[n] = x[n] - x[n-6]$$

Design of a Notch Filter









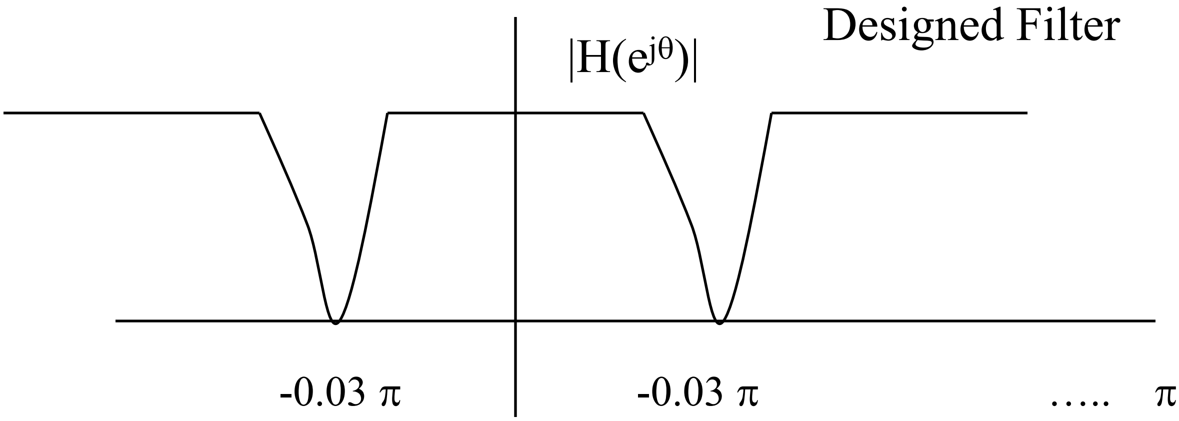
Pole - Zero Locations

Follows : $|H(e^{\pm j0.03\pi})| = 0$

and

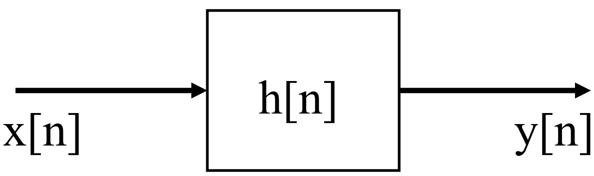
$$|H(e^{j\theta})| \approx 1$$
$$for$$
$$\theta \neq \pm 0.03\pi$$

Designed Filter



X

Sinusoidal Steady state response of discrete - time systems :



Convolution : $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$

Input x[n]: Complex exponential $e^{jn\theta}$

Output :

$$y_{ss}[n] = \sum_{m=-\infty}^{\infty} h[m].e^{j\theta(n-m)}$$

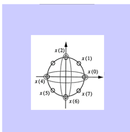
$$y_{ss}[n] = \sum_{m=-\infty}^{\infty} h[m].e^{j\theta n}e^{-j\theta m}$$

$$y_{ss}[n] = H(e^{j\theta})e^{j\theta n}$$

Where :

$$H(e^{j\theta}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j\theta m}$$

Note: **$\theta = wT$ (rad)**



$$H(e^{j\theta}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j\theta m}$$

$H(e^{j\theta})$ is the frequency response function.

Example : $h[n] = \delta[n] + \delta[n-1]$

$$H(e^{j\theta}) = 1 + e^{-j\theta}$$

Now consider the input to the system to be :

$$x[n] = 1 + 10\cos(\frac{n\pi}{10})$$

Re-writing the input in terms of complex exponentials:

$$x[n] = 1.e^{j0n} + 5e^{j\frac{n\pi}{10}} + 5e^{-j\frac{n\pi}{10}}$$

$$-\infty \leq n \leq \infty$$

$$x[n] = x_1[n] + x_2[n] + x_3[n]$$

Using the Superposition principle :

We have :

$$y_{ss}[n] = H(e^{j\theta_1})c_1 e^{j\theta_1} + H(e^{j\theta_2})c_2 e^{j\theta_2} + H(e^{j\theta_3})c_3 e^{j\theta_3}$$

Where :

$$H(e^{j\theta_1}) = H(e^{j0}) = 1 + e^{-j0} = 2$$

$$H(e^{j\theta_2}) = H(e^{j\frac{\pi}{10}}) = 1 + e^{-j\frac{\pi}{10}} = 1.97e^{-j0.16}$$

$$H(e^{j\theta_3}) = H(e^{-j\frac{\pi}{10}}) = 1 + e^{j\frac{\pi}{10}} = 1.97e^{j0.16}$$

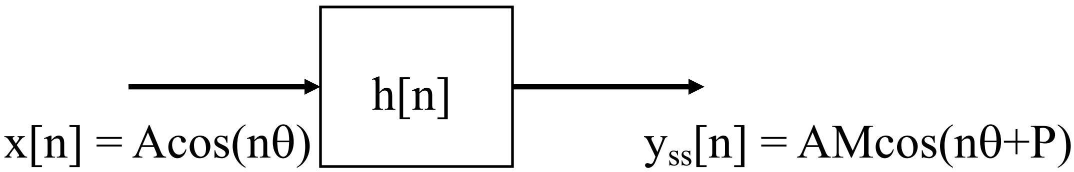
We have $c_1 = 1$, $c_2 = 5$, $c_3 = 5$

$$y_{ss}[n] = \sum_{k=1}^{3} H(e^{j\theta k})c_k e^{j\theta_k n}$$

$$y_{ss}[n] = 2 + 5.(1.97)\left[e^{j(\frac{n\pi}{10} - 0.16)} + e^{-j(\frac{n\pi}{10} - 0.16)} \right]$$

$$y_{ss}[n] = 2 + 19.7\cos(\frac{n\pi}{10} - 0.16)$$

Generalization:



$$H(e^{j\theta}) = \sum_{m=-\infty}^{\infty} h[m].e^{-j\theta m} = Me^{jP}$$

$$M = |H(e^{j\theta})|$$

$$P = \langle H(e^{j\theta})$$

Example :

$$h[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-2]$$

$$H(z) = 2 - 3z^{-1} + 4z^{-2}$$

$$H(e^{j\theta}) = 2 - 3e^{-j\theta} + 4e^{-j2\theta}$$

Example:

$$y[n] + 0.25y[n-4] = x[n] - x[n-2]$$

$$H(z) = \frac{z^2(z^2-1)}{z^4+0.25}$$

Filter Poles:

$$z^4 = -0.25$$

$$z^4 = 0.25e^{j(\pi+m2\pi)}$$

$$(z^4)^{1/4} = (0.25)^{1/4} e^{j(\pi+m2\pi)/4}$$

Poles are:

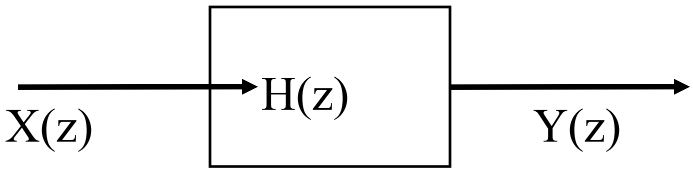
$$p_1 = 0.707e^{j\pi/4} \qquad p_3 = 0.707e^{j5\pi/4}$$

$$p_2 = 0.707e^{j3\pi/4} \qquad p_4 = 0.707e^{j7\pi/4}$$

Filter is stable :

$$H(e^{j\theta}) = \frac{e^{j2\theta}(e^{j2\theta}-1)}{e^{j4\theta}+0.25}$$

Only Stable systems have Sinusoidal Steady State Response.



$$Y(z) = H(z)X(z)$$

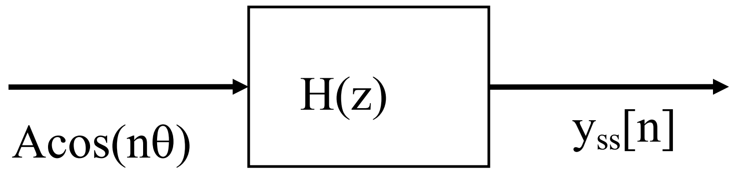
$$Y(z) = \frac{C_1 z}{z - p_1} + \frac{C_2 z}{z - p_2} + \underbrace{\frac{C_{in}}{z - e^{j\theta}} + \frac{C^*_{in}}{z - e^{-j\theta}}}_{\text{Sinusoidal Input}}$$

$$y[n] = \underbrace{C_1 P_1^{\,n} + C_2 P_2^{\,n}}_{\text{System Poles}} + \underbrace{C_{in} e^{j\theta n} + C^*_{in} e^{-j\theta n}}_{\text{Input Poles}}$$

$$y[n] = y_{ss}[n] = C_{in} e^{j\theta n} + C^*_{in} e^{-j\theta n}$$

if $|P_i| < 1$ (stable)

Generalization for Stable System



$$y_{ss}[n] = A|H(e^{j\theta})|\cos(n\theta + \angle H(e^{j\theta}))$$

where $H(e^{j\theta}) = H(z)$ for $z = e^{j\theta}$

Example :

A Digital Filter :

$$H(z) = \frac{z^2 - 0.2z - 0.08}{z^2 - 0.25}$$

$$|z| > 0.5$$

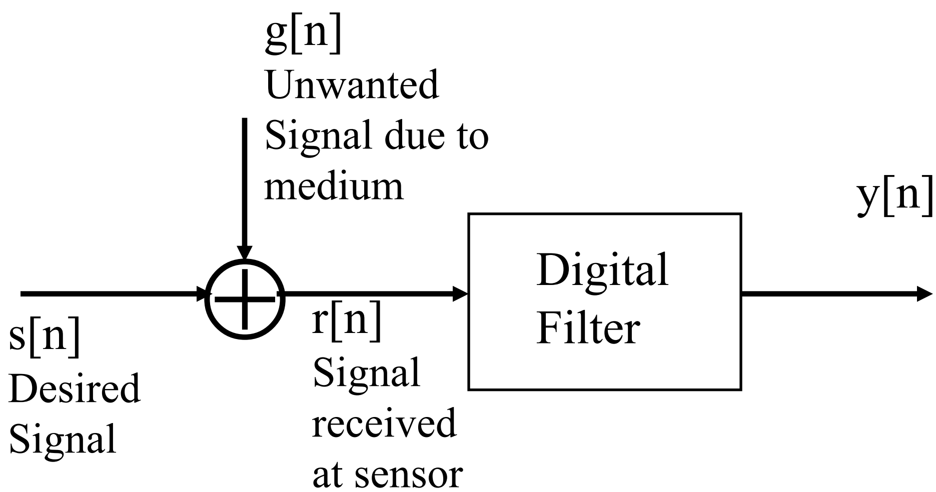
Poles / Zeros : poles: ±0.5   zeros: 0.4, -0.2

$\theta = wT = 2\pi fT = 2\pi f/f_s$

if $f = f_s/4$, $\theta = \pi/2$

$H(e^{j\pi/2}) = H(j) = 0.88e^{j0.18}$

A Typical Signal Processing Example :



- The purpose of designing a digital filter is to minimize the effect of the unwanted signal or to completely eliminate it from y[n].

- Specify for illustration :

$$s[n] = A\cos(n\theta) = 10\cos(\pi n/20)$$

$$g[n] = B\cos(10\theta n + \phi) = 4\cos(\pi n/2 + \pi/6)$$

$$r[n] = 10\cos(\pi n/20) + 4\cos(\pi n/2 + \pi/6)$$

Propose :

y[n] = r[n] + 0.9r[n-2]

Then :

$$H(z) = 1 + 0.9z^{-2}$$

$$H(e^{j\theta}) = 1 + 0.9e^{-j2\theta}$$

By Superposition :

For an Input of :

$$r[n] = 10\cos(\frac{\pi n}{20})$$

$$H(e^{j\frac{\pi}{20}}) = 1 + 0.9e^{-j\frac{\pi}{10}} = 1.88e^{-j0.15}$$

$$y_{1ss}[n] = 18.8\cos(\frac{\pi n}{20} - 0.15)$$

For an input of :

$$r[n] = 4\cos(\frac{\pi n}{2} + \frac{\pi}{6})$$

$$H(e^{j\frac{\pi}{2}}) = 0.1e^{j0}$$

$$y_{1ss}[n] = 0.4\cos(\frac{\pi n}{2} + \frac{\pi}{6})$$

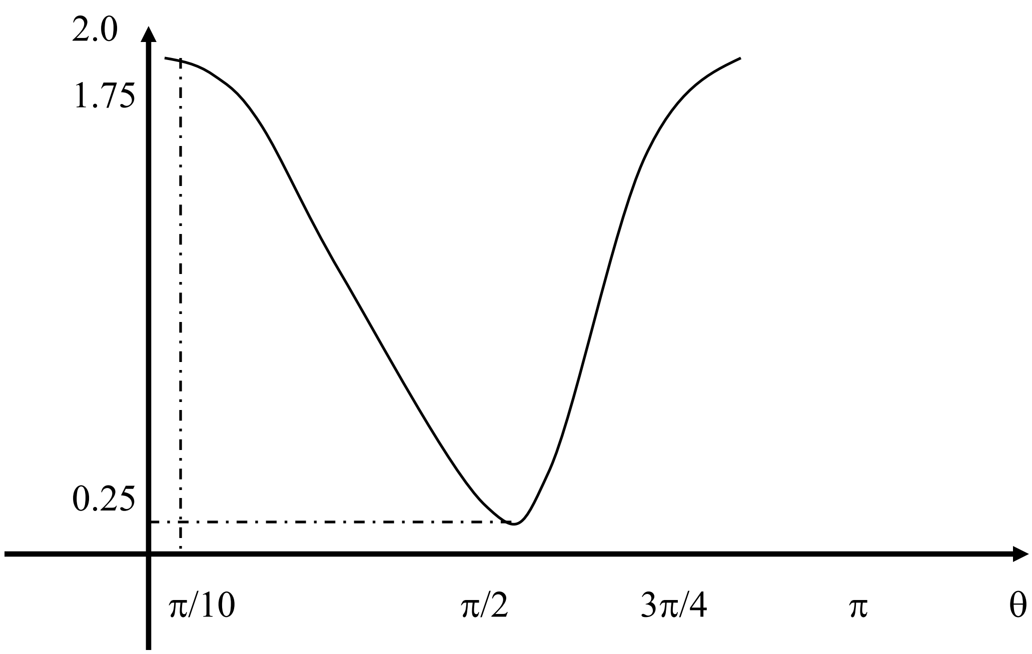
$$\therefore \Rightarrow$$

$$y_{ss}[n] = 18.8\cos(\frac{\pi n}{20} - 0.15) + 0.4\cos(\frac{\pi n}{2} + \frac{\pi}{6})$$

Notes:

1.) Desired Signal is amplified by a factor of 1.88 when passed through the filter.

2.) Filter response :



3.) Sketch the Pole zero locations of the filter.